

Design of a Fifth Order Butterworth Lowpass Filter

By Christopher Stahl, 2005

Abstract

An active implementation of a fifth order maximally flat low pass filter using Chris's Favored Biquad is presented.

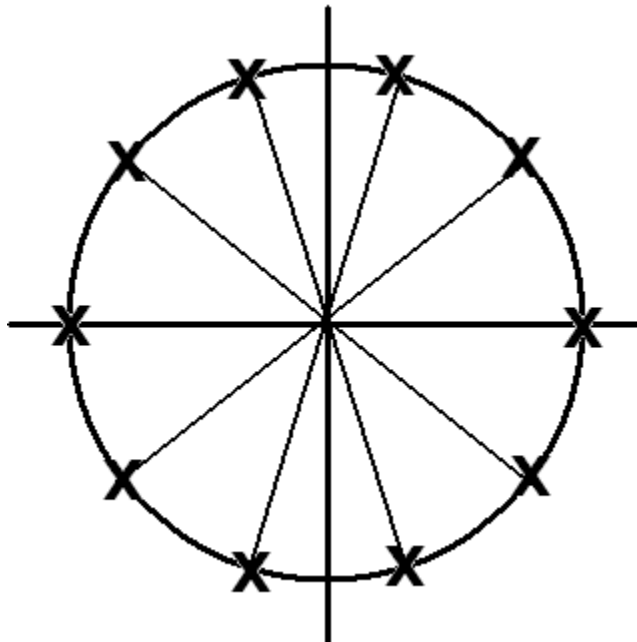
Background

A Butterworth filter is one that is optimized to have the flattest possible passband and ignores phase response and steepness of slope at cutoff frequency. The Bessel filter (named after the Bessel Polynomial) has the maximally flat phase response, while the Chebychev (named after the Chebychev Polynomial) has maximum slope at cutoff frequency. The Butterworth transfer function can be found to be:

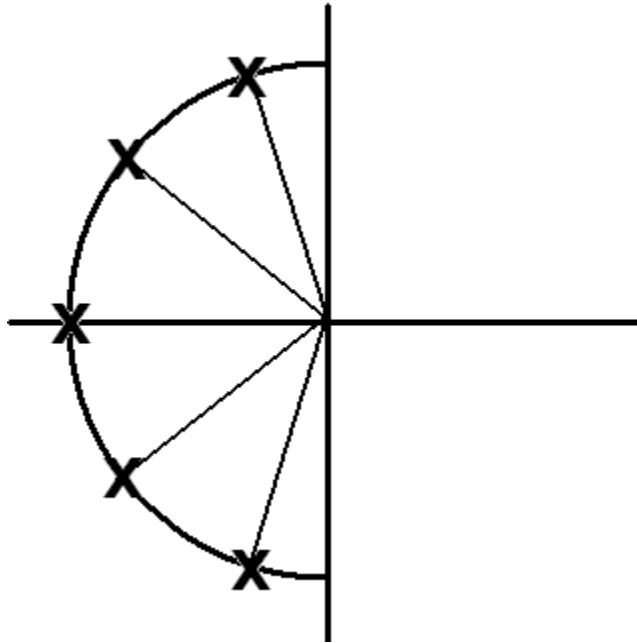
$$|H|^2 = \frac{1}{1 + \epsilon^2 s^{2n}}$$

Design

In my design, the cutoff attenuation was 3.01dB, which is known to correspond to $\epsilon = 1$. This means that for a normalized filter, all poles will be evenly spaced on a unit circle. For a filter of 5th order, this means 10 poles each placed 36 degrees apart. This can be shown graphically:

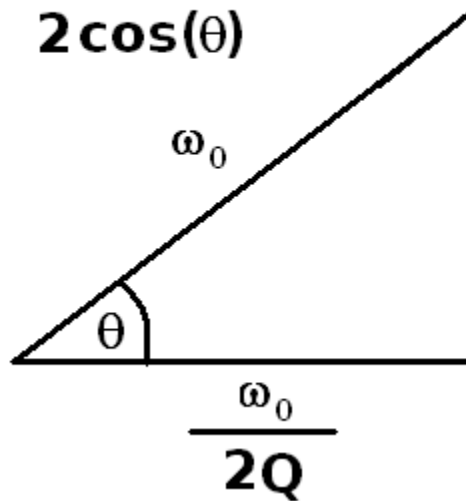


From this we know that the only valid system poles for increasing time systems would be the poles on the left half of the plane, as shown here:



From here, the coordinates of the system poles can be found on the ω_0, Q axis. This can be done using the relationship given by:

$$Q = \frac{1}{2 \cos(\theta)}$$



For my design, I had values of θ that were $0^\circ, 36^\circ, 72^\circ, -36^\circ,$ and -72° . For a normalized transfer function this gives Q values of 0.5, 0.618, and 1.618. The latter two Q values are for 2 pole systems, and the first Q is for a single pole system. This gives my filter a transfer function of:

$$A_v = \frac{\omega_0}{s + \omega_0} \frac{\omega_0^2}{s^2 + s \frac{\omega_0}{Q_2} + \omega_0^2} \frac{\omega_0^2}{s^2 + s \frac{\omega_0}{Q_3} + \omega_0^2}$$

The normalized Transfer function is:

$$A_v = \frac{1}{(s+1)} \frac{1}{(s^2 + s 1.618 + 1)} \frac{1}{(s^2 + s 0.618 + 1)}$$

Given: $Q_2=0.618, Q_3=1.618, \omega_c=2\pi 7241 \text{ Hz} = 45496 \frac{\text{rad}}{\text{s}}$

Transforming using $\omega_0 \rightarrow \frac{\omega_0}{\omega_c}$

$$A_v = \frac{2.198 \cdot 10^{-5}}{(s + 2.198 \cdot 10^{-5})} \frac{4.832 \cdot 10^{-10}}{(s^2 + s 3.556 \cdot 10^{-5} + 4.832 \cdot 10^{-10})} \frac{4.832 \cdot 10^{-10}}{(s^2 + s 1.358 \cdot 10^{-5} + 4.832 \cdot 10^{-10})}$$

Now all is needed is active implementations of first and second order systems. The first order system can be done using a RC network run off of the last filter stage.

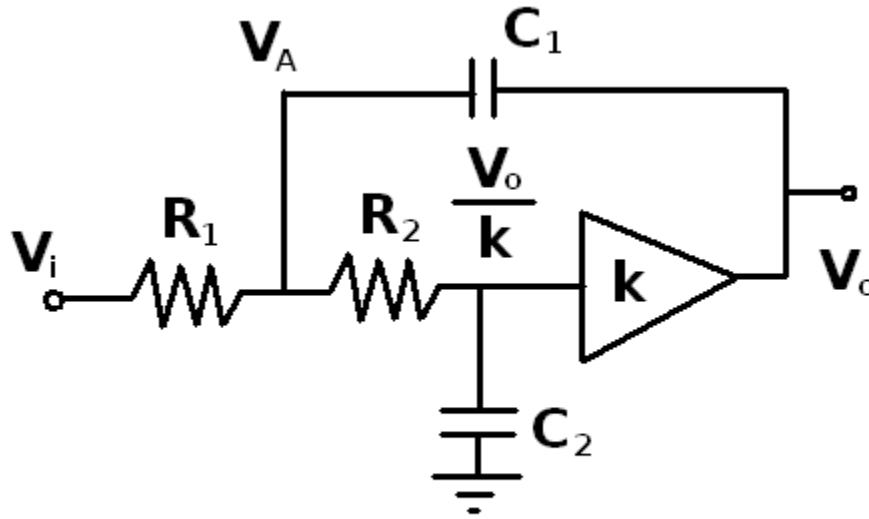
The second order stages can be built using Chris's Favored Biquads. CFB's are interesting circuits because they allow for ease of building. By adding two resistors of equal value, the gain can be made equal to 2 with good certainty. Further, this brings the opamp a little bit away from unity gain and thus further from oscillations due to opamp imperfections. By doing such, the capacitors can be made equal value, which allows for ease of hand matching, lower cost, and easy construction. At this point resistors can be chosen. Q can easily be set by setting the resistors in the circuit which vary predictably with Q.

A popular alternative is the unity gain Sallen-Key filter also shown in this report. It is more difficult to implement in practice because of unequal capacitor values. With unity gain, the quality of the circuit is set by the ratio of the capacitors; but high qualities are hard to obtain.

The final stage is a RC network where an additional resistors is added to the circuit to add attenuation to make up for the gain of 4 in this circuit. Such a feature may not be needed and from a noise perspective it is better not to attenuate the signal.

The appended pages at the end of this document will show the specific circuit as well as some results required by the assignment.

Analysis of Chris's Favored Biquad:



Assume:

$$\alpha = k^{-1}, G_1 = R_1^{-1}, G_2 = R_2^{-1}$$

Step 1: Do KCL of node V_A .

$$(V_A - V_i)G_1 + (V_A - \alpha V_o)G_2 + (V_A - V_o)sC_1 = 0$$

$$V_A(G_1 + G_2 + sC_2) - V_o(\alpha G_2 + sC_1) = V_i(G_1)$$

Step 2: Do KCL of node αV_o .

$$\alpha V_o s C_2 + (\alpha V_o - V_A)G_2 = 0$$

$$V_A = \alpha V_o \frac{(sC_2 + G_2)}{G_2} = V_o(sC_2 R_2 \alpha + \alpha)$$

Step 3: Substitute.

$$V_o(s^2 C_1 C_2 R_1 R_2 \alpha + s(C_2 R_2 G_1 \alpha + C_2 \alpha + s C_1 \alpha - s C_1) + (\alpha G_1)) = V_i G_1$$

Step 4: Find Transfer Function

$$A_v = \frac{1}{s^2 X + s Y + Z} \quad \text{where: } X = C_1 C_2 R_2 R_1 \alpha, Y = C_2 R_2 \alpha + C_2 R_1 \alpha + C_1 R_1 \alpha - C_1 R_1, Z = \alpha$$

Step 5: Put in a better form:

$$N = \frac{A_{v0}}{s^2 + s \frac{Y}{X} + \frac{Z}{X}} = \frac{\frac{1}{X}}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2}$$

let: $C_1 = C_2 = C, R_1 = a R, R_2 = a^{-1} R$

Step 6: Find Values

$$\omega_0^2 = \frac{Z}{X} = \frac{1}{C^2 R^2} \quad \text{and} \quad \omega_0 = \frac{1}{C R}$$

$$\frac{\omega_0}{Q} = \frac{Y}{X} \quad \text{thus} \quad Q = X \frac{\omega_0}{Y} = \frac{\alpha}{a^{-1} \alpha + 2 a \alpha - a}$$

let: $\alpha = 0.5, k = 2$

$$Q = \frac{0.5}{a^{-1} 0.5 + 2(0.5)a - a} = \frac{1}{a^{-1}} = a$$

$$N = \frac{2}{R^2 C^2}$$

Step 7: Assume Normalization based upon capacitance.

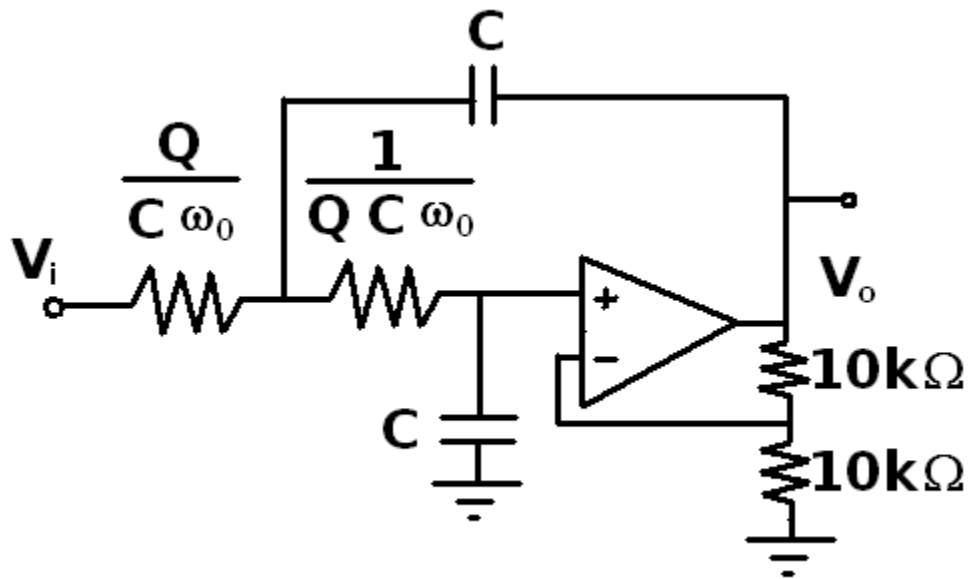
$$\omega_0 = 1, C = 1, R = \frac{1}{C \omega_0} = 1$$

if $\omega_c \rightarrow B \omega_0, C_c \rightarrow A \cdot C$ then $R' = \frac{1}{A \cdot B}$

Results:

$$A_v = \frac{N}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2} \quad \text{where} \quad \omega_0 = \frac{1}{RC}, Q = Q, N = 2 \omega_0^2 \quad \text{thus} \quad A_{v0} = 2 \frac{\omega_0^2}{\omega_0^2} = 2$$

$$A_v = \frac{\frac{2}{R^2 C^2}}{s^2 + s \frac{1}{QRC} + \frac{1}{R^2 C^2}}$$



Argument Against the Unity Gain Sallen-Key Filters:

Steps 1 – 3 same as above.

Step 4: Find Transfer Function

$$A_v = \frac{1}{s^2 X + s Y + Z} \quad \text{where: } X = C_1 C_2 R_2 R_1 \alpha, Y = C_2 R_2 \alpha + C_2 R_1 \alpha C_1 R_1 \alpha - C_1 R_1, Z = \alpha$$

Step 5: Put in a better form:

$$N = \frac{A_{v0}}{s^2 + s \frac{Y}{X} + \frac{Z}{X}} = \frac{\frac{1}{X}}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2}$$

Sallen-Key filters are defined with a unity gain.

$$\text{let: } \alpha = 1, C_1 = b C, C_2 = b^{-1} C, R_1 = a R, R_2 = a^{-1} R \quad X = C^2 R^2, Y = a^{-1} b^{-1} C R + a b^{-1} C R, Z = 1$$

Step 6: Find Values

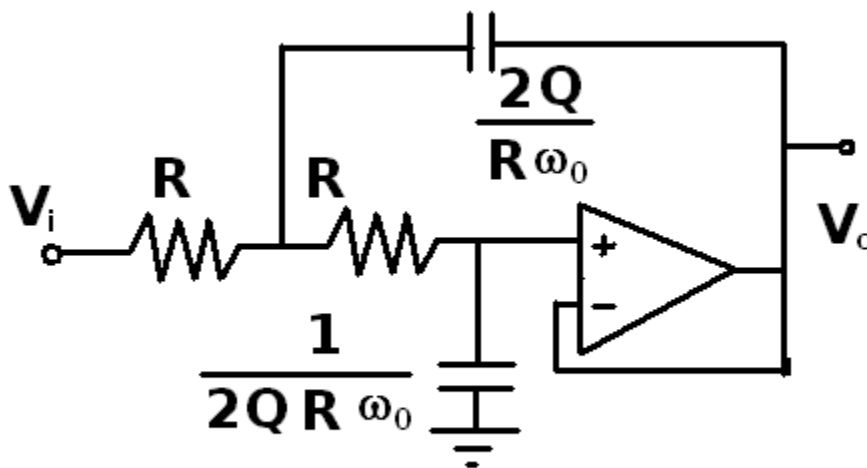
$$\omega_0^2 = \frac{Z}{X} = \frac{1}{C^2 R^2} \quad \text{and} \quad \omega_0 = \frac{1}{C R}$$

$$\frac{\omega_0}{Q} = \frac{Y}{X} \quad \text{thus} \quad Q = X \frac{\omega_0}{Y} = \frac{b}{a^{-1} + a}$$

In this case, it is easy to show that the Q of the filter cannot be controlled as easily by changing the resistors that have factors based upon a. With equal valued resistors, the Q is controlled by the capacitors which is a non-ideal situation. For the equal capacitor, unity gain case, it can be shown that Q never greater than 0.5, and thus only real poles exist.

Step 7: Results

$$Q = \frac{b}{2}, b = 2Q$$



Final Stage Analysis

Let: $\alpha = \frac{1}{A_v}, G_1 = \frac{1}{R_1}, G_2 = \frac{1}{R_2}$

Step 1: KCL at V_o .

$$V_o G_2 + V_o s C + V_o G_1 - V_i G_1 = 0$$

$$V_i (G_1) = V_o (G_2 + G_1 + s C)$$

Step 2: Find transfer function:

$$A_v = \frac{G_1}{G_2 + s C + G_1} = \frac{\frac{G_1}{G_1 + G_2}}{s + \left(\frac{G_2}{C} + \frac{G_1}{C}\right)} = \frac{\frac{G_1}{G_1 + G_2}}{s + \frac{1}{R_1 \parallel R_2 C}}$$

$$A_{v0} = \frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{\frac{1}{R_1}}{\frac{R_1 + R_2}{R_1 R_2}} = \frac{R_2}{R_1 + R_2}$$

$$\omega_0 = \frac{1}{(R_1 \parallel R_2) C}$$

Step 3: Create ratios for filter:

Let: $R_2 = a, R_1 = a(\alpha - 1)$ from $A_{v0} = \frac{a}{a\alpha - a + a} = \frac{1}{\alpha}$

Step 4: Normalize by capacitance:

$$R_1 \parallel R_2 = \frac{a(\alpha - 1 + 1)}{a^2(\alpha - 1)} = \frac{\alpha}{a(\alpha - 1)}$$

$$\omega_0 = \frac{1}{(R_1 \parallel R_2) C} = \frac{a(\alpha - 1)}{\alpha C}$$

$$a = \frac{\alpha}{(\alpha - 1) \omega_0 C}$$

thus: $R_2 = \frac{\alpha}{(\alpha - 1) \omega_0 C}, R_1 = \frac{\alpha}{\omega_0 C}$

